TWO CARDINALS MODELS WITH GAP ONE REVISITED

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Abstract. We succeed to say something on the identities of (μ^+, μ) when $\mu > \theta > \mathrm{cf}(\mu), \mu$ strong limit θ-compact or even μ limit of compact cardinals. This hopefully will help to prove that

(a) the pair (μ^+, μ) is compact and

(b) the consistency of "some pair (μ^+, μ) is not compact", however, this has not been proved.

The author would like to thank the Israel Science Foundation for partial support of this research (Grant No. 242/03). Publication 824.
I would like to thank Alice Leonhardt for the beautiful typing.

Annotated Content

§0 Introduction

[We give the basic definitions.]

§1 2-simplicity for gap one

[We prove that if $\mu = 2^{<\mu}$ then the family of identities of (μ^+, μ) is 2-simple. So this applies to μ singular strong limit but also, e.g., to triples $(\mu^+, \mu, \kappa), \mu = 2^{<\mu} > \kappa$.]

§2 Successor of strong limit above supercompact:2-identities

[Consider a pair (μ^+, μ) with μ strong limit singular $> \theta >$ $\mathrm{cf}(\mu), \theta$ a compact cardinal. We point out quite simply 2-identities which belong to $\mathrm{ID}_2(\mu^+, \mu)$ but not to $\mathrm{ID}_2(\aleph_1, \aleph_0)$.]

§0 Introduction

There has been much work on κ-compactness of pairs (λ,μ) of cardinals, i.e., when: if T is a set of first order sentences of cardinality $\le \kappa$ and every finite subset has a (λ,μ) mod M (i.e., $\|M\| = \lambda, |P^M| = \mu$ for a fixed unary P). Then T has a (λ,μ)-model.

A particularly important case is $\lambda = \mu^+$ in which case this can be represented as a problem on the κ-compactness of the logic $\mathbb{L}(\mathbf{Q}^{\rm card}_\lambda)$, i.e., $(\mathbf{Q}^{\rm card}_{\ge\lambda} x)\varphi$ says that there are at least λ element x satisfying φ_i. We deal here only with this case. See Furkhen [Fu65], Morley and Vaught [MoVa62], Keisler [Ke70], Mitchel [M1]; for more history see [Sh 604].

Now two cardinal theorems can be translated to partition problems: see [Sh 8], [Sh:E17], lately Shelah and Vaananan [ShVa 790].

Restricting ourselves to pairs (μ^+,μ), the identities of $(\aleph_1,\aleph_0)$ were sorted out in [Sh 74], but we do not know of the identities of any really different pair (μ^+,μ), i.e., one for which $(\aleph_1,\aleph_0) \nrightarrow (\mu^+,\mu)$. We know of some such pairs is suitable set theory. By Mitchel $(\aleph_2,\aleph_1)$ after suitably collapsing of a Mahlo strongly inaccessible to $\aleph_2$. The other, when there is a compact cardinal in $(\mathrm{cf}(\mu),\mu)$ by Litman and Shelah. So it would be nice to know (taking the extreme case).

0.1 Question: Assume μ is a singular cardinal the limit of compact and even supercompact cardinals.
1) What are the identities of (μ^+,μ)?
2) Is (μ^+,μ) $\aleph_0$-compact (equivalently μ-compact)?

Note that though we already know that there are some identities of (μ^+,μ) which are not identities of $(\aleph_1,\aleph_0)$ we have no explicit example. We give here a partial solution to 0.1(1) by finding families of such identities.

Another problem is consistency of failure of compactness.
In [Sh 604] we have dealt with the simplest case for pairs (λ,μ) by a reasonable criterion: including no use of large cardinals. From another perspective the simplest case is the consistency of non compactness of $\mathbb{L}(\mathbf{Q}), \mathbf{Q}$ one cardinality quantifier, and the simplest one is $\mathbf{Q} = \exists^{\ge\mu^+}$. So we are again drawn to pairs (μ^+,μ), that is gap one instead of gap 2 as in [Sh 604], so necessarily we need to use large cardinals as if, e.g., $\neg 0^\#$ then every such pair is compact.

0.2 Definition. 1) A partial identity[1] $\mathbf{s}$ is a pair $(a,e) = (\mathrm{Dom}_{\mathbf{s}}, e_{\mathbf{s}})$ where a is a finite set and e is an equivalence relation on a subfamily of the family of the finite subsets of a, having the property

[1] identification in the terminology of [Sh 8]

$$b\,e\,c \Rightarrow |b| = |c|.$$

The equivalence class of b with respect to e will be denoted b/e.
1A) We say $\mathbf{s}$ is a full identity or identity if $\mathrm{Dom}(e) = \mathscr{P}(a)$.
1B) We say that partial identities $\mathbf{s}_1 = (a_1, e_1), \mathbf{s}_2 = (a_2, e_2)$ are isomorphic if there is an isomorphism h from $\mathbf{s}_1$ onto $\mathbf{s}_2$ which mean that h is a one-to-one function from a_1 onto a_2 such that for every $b_1, c_1 \subseteq a_1$ we have $(b_1 e_1 c_1) \equiv h(b_1) e_2 h(b_2)$ (so h maps $\mathrm{Dom}(e_1)$ onto $\mathrm{Dom}(e_2)$). We define similarly "h is an embedding of $\mathbf{s}_1$ into $\mathbf{s}_2$.
2) We say that $\lambda \rightarrow (a, e)_\mu$, $\underline{\text{if}}$ (a, e) is an identity or a partial identity and for every function $f : [\lambda]^{<\aleph_0} \rightarrow \mu$, there is a one-to-one function $h : a \rightarrow \lambda$ such that

$$b\,e\,c \Rightarrow f(h''(b)) = f(h''(c)).$$

(Instead $\mathrm{Rang}(f) \subseteq \mu$ we may just require $|\mathrm{Rang}(f)| \leq \mu$, this is equivalent).
3) We define

$$\mathrm{ID}(\lambda, \mu) =: \{(n, e) : n < \omega \ \ \& \ \ (n, e) \text{ is an identity and } \lambda \rightarrow (n, e)_\mu\}$$

and for $f : [\lambda]^{<\aleph_0} \rightarrow X$ we let

$$\begin{aligned} \mathrm{ID}(f) =: \{(n, e) :& (n, e) \text{ is an identity such that for some one-to-one function} \\ & h \text{ from } n = \{0, \ldots, n-1\} \text{ to } \lambda \text{ we have} \\ & (\forall b, c \subseteq n)(b\,e\,c \Rightarrow f(h''(b)) = f(h''(c)))\}. \end{aligned}$$

Clearly two-place functions are easier to understand; this motivates:

0.3 Definition. 1) A two-identity or 2-identity[2] is a pair (a, e) where a is a finite set and e is an equivalence relation on $[a]^2$. Let $\lambda \rightarrow (a, e)_\mu$ mean $\lambda \rightarrow (a, e^+)_\mu$ where $be^+c \leftrightarrow [(bec) \vee (b = c \subseteq a)]$ for any $b, c \subseteq a$.
2) We defined

$$\mathrm{ID}_2(\lambda, \mu) =: \{(n, e) : \ (n, e) \text{ is a 2-identity and } \lambda \rightarrow (n, e)_\mu\}$$

we define $\mathrm{ID}_2(f)$ when $f : [\lambda]^2 \rightarrow X$ as

[2] it is not an identity as e is an equivalence relation on too small set but it is a partial identity

$$\Big\{(n,e) : (n,e) \text{ is a two-identity such that for some } h,$$
$$\text{a one-to-one function from } \{0,\dots,n-1\} \text{ into } \lambda$$
$$\text{we have } \{\ell_1,\ell_2\}e\{k_1,k_2\} \text{ implies that } \ell_1 \neq \ell_2 \in \{0,\dots,n-1\},$$
$$k_1 \neq k_2 \in \{0,\dots,n-1\} \text{ and } f(\{h(\ell_1),h(\ell_2)\}) = f(\{h(k_1),h(k_2)\})\Big\}.$$

3) Let us define

$$\begin{aligned}\mathrm{ID}_2^{\circledast} =: \{({}^n2,e) : \ & ({}^n2,e) \text{ is a two-identity and if} \\ & \{\eta_1,\eta_2\} \neq \{\nu_1,\nu_2\} \text{ are } \subseteq {}^n2, \text{ then} \\ & \{\eta_1,\eta_2\}e\{\nu_1,\nu_2\} \Rightarrow \eta_1 \cap \eta_2 = \nu_1 \cap \nu_2\}.\end{aligned}$$

4) In parts (1) and (2) we may replace 2 by $k < \omega$ (only $k < |a_{\mathbf{s}}|$ is interesting) and by $(\leq k)$.

<u>0.4 Discussion</u>: By [Sh 49], under the assumption $\aleph_\omega < 2^{\aleph_0}$, the families $\mathrm{ID}_2(\aleph_\omega,\aleph_0)$ and $ID_2^{\circledast}$ coincide (up to an isomorphism of identities). In Gilchrist and Shelah [GcSh 491] and [GcSh 583] we considered the question of the equality between these $\mathrm{ID}_2(2^{\aleph_0},\aleph_0)$ and $\mathrm{ID}_2^{\circledast}$ under the assumption $2^{\aleph_0} = \aleph_2$. We showed that consistently the answer may be "yes" and may be "no".

Note that $(\aleph_n,\aleph_0) \nrightarrow (\aleph_\omega,\aleph_0)$ so $\mathrm{ID}(\aleph_2,\aleph_0) \neq \mathrm{ID}(\aleph_\omega,\aleph_0)$, but for identities for pairs (i.e. ID_2) the question is meaningful.

We can look more at ordered identities

0.5 Definition. 1) An ord-identity or order identity is an identity $\mathbf{s}$ such that $a_s \subseteq$ Ord or just: a is an ordered set.
2) $\lambda \rightarrow_{or} (\mathbf{s})_\mu$ if $\mathbf{s}$ is an ord-identity and for every $\mathbf{c} : [\lambda]^{<\aleph_0} \rightarrow \mu$ we have $\mathbf{s} \in \mathrm{OID}(\mathbf{c})$, see below (equivalently $\mathrm{Dom}(\mathbf{c}) = [\lambda]^{<\aleph_0}, |\mathrm{Rang}(\mathbf{c})| \leq \mu$).
3) For $\mathbf{c} : [\lambda]^{<\aleph_0} < \mu$ let $\mathrm{OID}(\mathbf{c}) = \{(a,e) : a$ is a set of ordinals and there is an order preserving function $f : a \rightarrow \lambda$ such that $b_1 e b_2 \Rightarrow \mathbf{c}(f''(b_1)) = \mathbf{c}(f''(b_2))\}$.
4) $\mathrm{OID}(\lambda,\mu) = \{(n,e) : (n,e) \in \mathrm{OID}(\mathbf{c})$ for every $\mathbf{c} : [\lambda]^{<\aleph_0} \rightarrow \mu$ we say $(n,e) \in \mathrm{OID}(\mathbf{c})\}$.
5) Similarly OID_2, OID_k, $\mathrm{OID}_{\leq k}$.

Of course,

0.6 Claim. *1)* $\mathrm{ID}(\lambda,\mu)$ *can be computed from* $\mathrm{OID}(\lambda,\mu)$.
2) Let a be a finite set of ordinals and e a function. If (a,e) is an identity, a a set of ordinals and $\lambda>\mu$, <u>then</u> $(a,e)\in \mathrm{ID}(\lambda,\mu)$ <u>iff</u> for some permutation π of a we have $(a,e^\pi)\in \mathrm{OID}(\lambda,\mu)$ where $e^\pi=\{(b,c):(\pi''(b),\pi''(c))\in e\}$.
3) Let A be a set of ordinals, (a,e) an ord-identity and $\mathbf{c}$ a function with domain $[A]^{<\aleph_0}$. <u>Then</u> $(a,e)\in \mathrm{ID}(\mathbf{c})$ <u>iff</u> for some permutation π of $a,(a,e^\pi)\in \mathrm{OID}(\mathbf{c})$.
4) Similarly for 2-identities and k-identities and $(\le k)$-identities and partial identities.

0.7 Claim. *For $n\in[1,\omega)$ and $\mathbf{s}$ an ordered partial identity <u>then</u> there is a first order sentence $\psi_{\mathbf{s}}$ such that: $\psi_{\mathbf{s}}$ has a (μ^{+n},μ)-model <u>iff</u> $\mathbf{s}\notin \mathrm{OID}(\mu^{+n},\mu)$.*

Proof. Easy as for some first order ψ sentence if M is a (μ^{+n},μ)-model of ψ <u>then</u> $<^M$ is a linear order of M (of cardinality μ^{+n}) which is μ^{+n}-like (i.e. every initial segment has cardinality). $\square_{0.7}$

We define simplicity:

0.8 Definition. 1) For $k\le\aleph_0$, we say (λ,μ) has k-simple identities <u>when</u> $(a,e)\in \mathrm{ID}(\lambda,\mu)\Rightarrow(a,e')\in \mathrm{ID}(\lambda,\mu)$ whenever:

$(*)_k$ $a\subseteq\omega,(a,e)$ is an identity of (λ,μ) and e' is defined by

$$be'c \text{ iff } |b|=|c| \ \ \& \ \ (\forall b'c')[b'\subseteq b \ \ \& \ \ |b'|\le k \ \ \& \ \ c'= \mathrm{OP}_{c,b}(b')\rightarrow b'ec];$$

$$\text{recall } \mathrm{OP}_{A,B}(\alpha)=\beta \text{ iff } \alpha\in A \ \ \& \ \ \beta\in B \ \ \& \ \ \mathrm{otp}(\alpha\cap A)= \mathrm{otp}(\beta\cap B).$$

2) We define "(λ,μ) for k-simple ordered identities".

We can ask
<u>0.9 Question</u>: 1) Define reasonably a pair (λ,μ) such that consistently

- $\circledast$ $\mathrm{ID}(\lambda,\mu)$ is not recursive
- $\circledast'$ $\mathrm{ID}(\lambda,\mu)$ is not, in a reasonable way, finitely generated.

2) Similarly for $\mathrm{ID}_2(\lambda,\mu)$.
3) Restrict yourself to (μ^+,μ).

§1 2-simplicity for gap one

1.1 Claim. *1) If μ is strong limit singular* *then* $\mathrm{ID}_2(\mu^+,\mu)$ *is 2-simple.*
2) If $\mu = 2^{<\mu}$ and $c_0 : [\mu^+]^{<\aleph_0} \to \mu$ then we can find $c^* : [\mu^+]^2 \to \mu$ such that:

- (α) *if $n \in [2,\omega)$ and $\alpha_0,\dots,\alpha_{n-1} < \mu^+$ are with no repetitions and $\beta_0,\dots,\beta_{n-1} < \mu^+$ are with no repetitions and $\ell < k < n \Rightarrow c^*\{\alpha_\ell,\alpha_k\} = c^*\{\beta_\ell,\beta_k\}$ then $c_0\{\alpha_0,\dots,\alpha_{n-1}\} = c_0\{\beta_0,\dots,\beta_{n-1}\}$ and even $c^*\{\alpha_0,\dots,\alpha_{n-1}\} = c^*\{\beta_0,\dots,\beta_{n-1}\}$*
- (β) *if in addition $\alpha_0 < \alpha_1 < \dots < \alpha_{n-1}$ then $\beta_0 < \beta_1 < \dots < \beta_{n-3} < \beta_{n-2}, \beta_{n-3} < \beta_{n-1}$.*

1.2 Remark. 1) We may wonder what is the gain in 1.1(2) as compared to 1.1(1), as if $\mu = 2^{<\mu}$ is regular then we know all relevant theory on (μ^+,μ)? The answer is that it clarifies identities of triples (μ^+,μ,κ), e.g.

- (a) $(\mu^+,\mu,\kappa), \mu$ strong limit singular $> \kappa \geq \mathrm{cf}(\mu)$
- (b) $(\mu^+,\mu,\kappa), \mu = \mu^{\beth_\omega(\kappa)}$.

2) Replacing $\mu^+, 2$ by $\mu^{+k}, k+1 \geq 2$ is similar and easier.

Proof. 1) By part (2).
2) By $\boxdot_1 - \boxdot_5$ below the claim is easy (see details in the end).

- $\boxdot_1$ There is $c_1 : [\mu^+]^2 \to \mu$ such that if $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$ and $\beta_0,\beta_1,\beta_2 < \mu^+$ are with no repetitions and $c_1\{\beta_\ell,\beta_k\} = c_1\{\alpha_\ell,\alpha_k\}$ for $\ell < k < 3$ then at least two of the following holds $\beta_0 < \beta_1, \beta_0 < \beta_2, \beta_1 < \beta_2$.

[Why? Let $\eta_\alpha \in {}^\mu 2$ for $\alpha < \mu^+$ be pairwise distinct and for $\alpha \neq \beta < \mu^+$ let $\varepsilon\{\alpha,\beta\} = \mathrm{Min}\{\varepsilon : \eta_\alpha \restriction \varepsilon \neq \eta_\beta \restriction \varepsilon\}$ and define the function c'_1 with domain $[\mu^+]^2$ by $c'_1\{\alpha,\beta\} = \{\eta_\alpha \restriction \varepsilon\{\alpha,\beta\}, \eta_\beta \restriction \varepsilon\{\alpha,\beta\}\}$, now $|\mathrm{Rang}(c'_1)| \leq \mu$ holds because $\mu = 2^{<\mu}$. For $\alpha \neq \beta$, let $c''_1\{\alpha,\beta\}$ be 1 if $(\eta_\alpha <_{\mathrm{lex}} \eta_\beta) \equiv (\alpha < \beta)$ and 0 otherwise (the Sierpinski colouring). Lastly, define c_1 by $c_1, c_1\{\alpha,\beta\} = (c'_1\{\alpha,\beta\}, c''_1\{\alpha,\beta\})$, it is a function with domain $[\mu^+]^2$ and range of cardinality $\leq \mu$ and easily it is as required.]

- $\boxdot_2$ for every $c : [\mu^+]^{<\aleph_0} \to \mu$ there is $c_2 : [\mu^+]^2 \to \mu$ such that: if $n \geq 2, \alpha_0 < \alpha_1 < \dots < \alpha_{n-1} < \mu^+, \beta_0 < \beta_1 < \dots < \beta_{n-1} < \mu^+$ and $\ell < k < n \Rightarrow c_2\{\alpha_\ell,\alpha_k\} = c_2\{\beta_\ell,\beta_k\}$ then $c\{\alpha_0,\dots,\alpha_{n-1}\} = c\{\beta_0,\dots,\beta_{n-1}\}$.

[Why? We are given $c : [\mu^+]^{<\aleph_0} \to \mu$ and for each $\alpha < \mu^+$ let f_α be a one-to-one function from α onto the ordinal $|\alpha| \le \mu$ and we shall use those f_α's also later. We define an equivalence relation E on $[\mu^+]^2$

- $(*)$ for $\alpha_1 < \beta_1 < \mu^+$ and $\alpha_2 < \beta_2 < \mu^+$ we have $\{\alpha_1, \beta_1\} E \{\alpha_2, \beta_2\}$ <u>iff</u>
  - (a) $f_{\beta_1}(\alpha_1) = f_{\beta_2}(\alpha_2)$ and
  - (b) for any $n < \omega$ and $\gamma_0 < \ldots < \gamma_{n-1} < f_{\beta_1}(\alpha_1)$ we have

$$c\{\alpha_1, \beta_1, f^{-1}_{\beta_1}(\gamma_0), \ldots, f^{-1}_{\beta_1}(\gamma_{n-1})\} = c\{\alpha_2, \beta_2, f^{-1}_{\beta_2}(\gamma_0), \ldots, f^{-1}_{\beta_1}(\gamma_{n-1})\}$$

  and similarly if we omit α_1, α_2 and/or β_1, β_2.

So $[\mu^+]^2/E$ has cardinality $\le {}^{\mu>}2 = \mu$ and let $c_2 : [\mu^+] \to \mu$ be such that $c_2\{\alpha_1, \beta_1\} = c_2\{\alpha_2, \beta_2\}$ iff $\{\alpha_1, \beta_1\}/E = \{\alpha_2, \beta_2\}/E$. We now check that it is as required in $\boxdot_2$. Let $n, \langle \alpha_\ell : \ell < n \rangle, \langle \beta_\ell : \ell < n \rangle$ be as in $\boxdot_2$; so $\ell < k < n \Rightarrow c_2\{\alpha_\ell, \alpha_n\} = c_2\{\beta_\ell, \beta_n\}$, hence by $(*)(a)$ above (for $k = n-1$) we have $\ell < n-1 \Rightarrow f_{\alpha_{n-1}}(\alpha_\ell) = f_{\beta_{n-1}}(\beta_\ell)$, call it γ_ℓ. Let $\ell(*) < n(*)$ be such that γ_ℓ is maximal. Now apply $(*)(b)$ with $\alpha_{\ell(*)}, \alpha_{n-1}, \beta_{\ell(*)}, \beta_{n-2}$ here standing for $\alpha_1, \beta_1, \alpha_2, \beta_2$ there and we get the desired result.]

$\boxdot_3$ In $\boxdot_2$, using $f_\alpha : \alpha \to \mu$ as in its proof, we have $c\{\alpha_0, \ldots, \alpha_{n-1}\} = c\{\beta_0, \ldots, \beta_{n-2}\}$ also when
- $(*)$ $n \ge 2, \alpha_0 < \alpha_1 < \ldots < \alpha_{n-3} < \alpha_{n-2} < \alpha_{n-1} < \mu^+, \beta_0 < \beta_1 < \ldots < \beta_{n-3} < \beta_{n-1} < \beta_{n-2}$ and $\ell < n-2 \Rightarrow f_{\alpha_{n-1}}(\alpha_\ell) = f_{\alpha_{n-2}}(\alpha_\ell)$ and $\ell < k < n \Rightarrow c_2\{\alpha_\ell, \alpha_k\} = c_2\{\beta_\ell, \beta_k\}$.

[Why? Just the same proof.]

$\boxdot_4$ there is $c_4 : [\mu^+] \to \mu$ such that if $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$ and $\beta_0, \beta_1, \beta_2 < \mu^+$ with no repetitions, $c_4\{\beta_\ell, \beta_k\} = c_4\{\alpha_\ell, \alpha_k\}$ for $\ell < k < 3$ then $\beta_0 < \beta_1$ & $\beta_0 < \beta_2$.

[Why? For $\alpha < \beta < \mu^+$ we let $c'\{\alpha, \beta\} = \{f_\beta(\gamma) : \gamma < \alpha \;\&\; f_\beta(\gamma) < f_\beta(\beta)\}$ and let $c_4\{\alpha, \beta\} = (c'\{\alpha, \beta\}, c_1\{\alpha, \beta\}, f_\beta(\alpha))$ where c_1 is from $\boxdot_1$ and $\langle f_\gamma : \gamma < \mu^+ \rangle$ is from the proof of $\boxdot_2$. Clearly $|\text{Rang}(c')| \le \sum\limits_{\zeta < \mu} 2^{|\zeta|} = \mu$ hence $|\text{Rang}(c_4)| \le \mu^3 = \mu$. If $\alpha_\ell, \beta_\ell (\ell < 3)$ form a counterexample, then $c_1\{\alpha_\ell, \alpha_k\} = c_1\{\beta_\ell, \beta_k\}$ for $\ell < k < 3$ hence by $\boxdot_1$ we have four cases according to which one of the inequalities $\beta_\ell < \beta_k, \ell < k < 3$ fail. So the proof of $\boxdot_4$ splits to three cases.

<u>Case 0</u>: $\beta_0 < \beta_1 < \beta_2$.

Trivial: the desired conclusion holds.

Case 1: $\beta_1 < \beta_0$ so $\beta_1 < \beta_0 < \beta_2$.

Let $\zeta_\ell = f_{\alpha_2}(\alpha_\ell)$ for $\ell = 0, 1$ hence $\zeta_0 \neq \zeta_1$ as f_{α_2} is one to one and $\zeta_\ell = f_{\beta_2}(\beta_\ell)$. Now on the one hand if $\zeta_0 < \zeta_1$ then $c'\{\alpha_1, \alpha_2\} \neq c'\{\beta_1, \beta_2\}$ (as $\zeta_0 \in c'\{\alpha_1, \alpha_2\}, \zeta_0 \notin c'\{\beta_1, \beta_2\}$), contradiction. On the other hand if $\zeta_1 < \zeta_0$ then $c'\{\alpha_0, \alpha_2\} \neq c'\{\beta_0, \beta_2\}$ (as $\zeta_1 \in c'\{\beta_0, \beta_2\}, \zeta_1 \notin c'\{\alpha_0, \alpha_2\}$), a contradiction, too.

Case 2: $\beta_2 < \beta_0$.

Then at least one of $\beta_1 < \beta_0, \beta_2 < \beta_1$ hold contradicting $\boxdot_1$, (i.e., the case we are in).

Case 3: $\beta_2 < \beta_1$.

By $\boxdot_1$ we have $\beta_0 < \beta_2 < \beta_1$.

This is O.K. for $\boxdot_4$.]

$\boxdot_5$ for every $c : [\mu^+]^2 \to \mu$ there is $c_5 : [\mu^+]^2 \to \mu$ such that

- (a) $c_5\{\alpha_1, \beta_1\} = c_5\{\alpha_2, \beta_2\} \Rightarrow c_2\{\alpha_1, \beta_1\} = c_2\{\alpha_2, \beta_2\}$ where c_2 is from $\boxdot_2$ (so also $\boxdot_3$)
- (b) there are no $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$ and $\beta_0 < \beta_1 < \beta_2 < \mu^+$ such that $f_{\alpha_2}(\alpha_0) \neq f_{\alpha_1}(\alpha_0), c_5\{\alpha_0, \alpha_1\} = c_5\{\beta_0, \beta_2\}, c_5\{\alpha_0, \alpha_2\} = c_5\{\beta_0, \beta_1\}$ and $c_5\{\alpha_1, \alpha_2\} = c_5\{\beta_1, \beta_2\}$
- (c) $c_5\{\alpha_1, \beta_1\} = c_5\{\alpha_2, \beta_2\} \Rightarrow c_4\{\alpha_1, \beta_1\} = c_4\{\alpha_2, \beta_2\}$ where c_4 is from $\boxdot_4$.

[Why? Let $\kappa = \mathrm{cf}(\mu) \leq \mu$ and $\mu = \sum\limits_{i<\kappa} \lambda_i$ be such that if μ is a limit cardinal then λ_i is (strictly) increasing continuous and if μ is a successor cardinal then $\mu = \lambda^+$ and $\lambda_i = \lambda$ for $i < \kappa$. We can find $d : [\mu^+]^2 \to \kappa$ and $\bar{g}$ such that

$\circledast_0$
- (i) for $\beta < \mu^+, i < \kappa$ the set $A_{\beta,i} =: \{\alpha < \beta : d\{\alpha, \beta\} \leq i\}$ has cardinality $\leq \lambda_i$ and
- (ii) if $\alpha < \beta < \gamma < \mu^+$ then $d\{\alpha, \gamma\} \leq \max\{d\{\alpha, \beta\}, d\{\beta, \gamma\}\}$
- (iii) $\bar{g}$ is a sequence $\langle g_\alpha : \alpha < \mu^+ \rangle$
- (iv) $g_\alpha : \alpha \to \mu$ is one to one and $\lambda_i^+ < \mu \ \& \ i < \kappa \ \& \ \alpha < \beta \Rightarrow ((g_\beta(\alpha) < \lambda_i^+) \equiv (d\{\alpha, \beta\} \leq i))$
- (v) if $\alpha < \beta, d\{\alpha, \beta\} = i$ and $\lambda_i^+ = \mu$ then $g_\beta(\alpha) < d\{\alpha, \beta\}$.

[Why we can find them? By induction on $\beta < \mu^+$ by induction on $i < \mu$ for $\alpha = f_\beta^{-1}(i)$ we choose $d\{\alpha,\beta\}$ and $g_\beta(\alpha)$ as required.]

Define the functions c_6' and c_7' with domain $[\mu^+]^2$ as follows: if $\alpha < \beta$ then $c_6'\{\alpha,\beta\} = \{(t,\zeta_0,\zeta_1) : \zeta_0,\zeta_1 \le g_\beta(\alpha), t < 2$ and $t = 0 \Rightarrow g_\beta^{-1}(\zeta_1) < g_\beta(\zeta_2), t = 1 \Rightarrow g_\beta(\zeta_1) > g_\beta(\zeta_2)\}$ and $c_7'\{\alpha,\beta\} = \{(t,\zeta,\xi) : \zeta \in \lambda^+_{d\{\alpha,\beta\}} \cap \mathrm{Rang}(g_\alpha)$ and $\xi \in \lambda^+_{d\{\alpha,\beta\}} \cap \mathrm{Rang}(g_\beta)$ and $[\lambda^+_{d\{\alpha,\beta\}} = \mu \Rightarrow \zeta < d\{\alpha,\beta\}\ \ \&\ \ \xi < d\{\alpha,\beta\}]$ and $g_\alpha^{-1}(\zeta) < g_\beta^{-1}(\xi)\ \ \&\ \ t = 0$ or $g_\alpha^{-1}(\zeta) = g_\beta^{-1}(\xi)\ \ \&\ \ t = 1$ or $g_\alpha^{-1}(\zeta) > g_\beta^{-1}(\xi)\ \ \&\ \ t = 2\}$.

Now for $\alpha < \beta < \mu^+$ we define $c_5'\{\alpha,\beta\} \in \Pi\{\lambda_j^+ : j \le d\{\alpha,\beta\}\}$, we do this by induction on β and for a fixed β by induction $i = d\{\alpha,\beta\}$ and for a fixed β and i by induction on α.

Arriving to $\alpha < \beta$ so $\zeta < \lambda^+_{d\{\alpha,\beta\}}$, for each $j \le d\{\alpha,\beta\}$, let $(c_5'\{\alpha,\beta\})(j)$ be the first ordinal $\xi < \lambda_j^+$ such that:

$\circledast_1$ if $\gamma < \beta\ \ \&\ \ d\{\gamma,\beta\} \le j\ \ \&\ \ (d\{\gamma,\beta\} = d\{\alpha,\beta\} \Rightarrow \gamma < \alpha)$ then

$$(c_5'\{\alpha,\gamma\})(j) < \xi.$$

Clearly possible. The colouring we use is c_5 where for $\alpha < \beta < \mu^+$ we let $c_5\{\alpha,\beta\} = (d\{\alpha,\beta\}, g_\beta(\alpha), f_\beta(\alpha), c_2\{\alpha,\beta\}, c_5'\{\alpha,\beta\}, c_6'\{\alpha,\beta\}, c_7'\{\alpha,\beta\}, c_4\{\alpha,\beta\})$, recalling c_4 is from $\boxdot_4$ and c_2 is from $\boxdot_2$. Obviously, $|\mathrm{Rang}(c_5)| \le \mu$ and clauses (a) + (c) of $\boxdot_5$ holds. So assume $\alpha_0 < \alpha_1 < \alpha_2, \beta_0 < \beta_1 < \beta_2$ form a counterexample to clause (b) of $\boxdot_5$ and we shall eventually derive a contradiction.

Clearly

$\circledast_2$ (i) $d\{\alpha_0,\alpha_2\} = d\{\beta_0,\beta_1\}, d\{\alpha_0,\alpha_1\} = d\{\beta_0,\beta_2\}, d\{\alpha_1,\alpha_2\} = d\{\beta_1,\beta_2\}$

(ii) similarly for c', c_0', c_1', c_4.

By clause (ii) above we have $d\{\alpha_0,\alpha_2\} \le \max\{d\{\alpha_0,\alpha_1\}, d\{\alpha_1,\alpha_2\}\}$, and applying clause (ii) to $\beta_0 < \beta_1 < \beta_2$ and using $\circledast_2$ we have $d\{\alpha_0,\alpha_1\} \le \max\{d\{\alpha_0,\alpha_2\}, d\{\alpha_1,\alpha_2\}$. Hence $d\{\alpha_0,\alpha_1\} = d\{\alpha_0,\alpha_2\} > d\{\alpha_1,\alpha_2\}$ or $\bigwedge\limits_{\ell=1}^{2}[d\{\alpha_0,\alpha_\ell\} \le d\{\alpha_1,\alpha_2\}]$; we deal with those two cases separately.

<u>Case 1</u>: $\varepsilon = d\{\alpha_0,\alpha_1\} = d\{\alpha_0,\alpha_2\} > d\{\alpha_1,\alpha_2\}$.

So (see the definition of c_5', with $\alpha_0,\alpha_2,\alpha_1,\varepsilon$ here standing for α,β,γ,j there recalling that $\alpha_0 < \alpha_1 < \alpha_2$) we have $\lambda_\varepsilon^+ > (c_5'\{\alpha_0,\alpha_2\})(\varepsilon) > (c_5'\{\alpha_0,\alpha_1\})(\varepsilon)$. Similarly, $\lambda_\varepsilon^+ > (c_5'\{\beta_0,\beta_2\})(\varepsilon) > (c_5'\{\beta_0,\beta_1\})(\varepsilon)$. This contradicts $c_5'\{\alpha_0,\alpha_\ell\} = c_5'\{\beta_0,\beta_{3-\ell}\}$ for $\ell = 1,2$.

Case 2: $d\{\alpha_0,\alpha_\ell\} \le d\{\alpha_1,\alpha_2\}$ for $\ell = 1,2$.

Let $\varepsilon = d\{\alpha_1,\alpha_2\}$. Let $\zeta_\ell = g_{\alpha_\ell}(\alpha_0)$ for $\ell = 1,2$ so $\zeta_\ell = g_{\beta_{3-\ell}}(\beta_0)$ for $\ell = 1,2$. By the assumption toward contradiction, i.e., by a demand in clause (b) of $\boxdot_5$ we have $\zeta_1 \ne \zeta_2$. Clearly $\zeta_\ell < \lambda^+_{d\{\alpha_0,\alpha_\ell\}} \le \lambda^+_{d\{\alpha_1,\alpha_2\}} = \lambda^+_\varepsilon$ and $\lambda^+_\varepsilon = \mu \Rightarrow \zeta_\ell < d\{\alpha_0,\alpha_\ell\} \le d\{\alpha_1,\alpha_2\} \le \varepsilon$.

As $c'_7\{\alpha_1,\alpha_2\} = c'_7\{\beta_1,\beta_2\}$ and $g^{-1}_{\alpha_1}(\zeta_1) = g^{-1}_{\alpha_2}(\zeta_2)$ clearly $g^{-1}_{\beta_1}(\zeta_1) = g^{-1}_{\beta_2}(\zeta_2)$ and they are well defined.

For $\ell = 1,2$ as $c_5\{\alpha_0,\alpha_\ell\} = c_5\{\beta_0,\beta_{3-\ell}\}$ by the choice of ζ_ℓ (that is $\zeta_\ell = g_{\alpha_\ell}(\alpha_0)$) we have $g_{\beta_\ell}(\beta_0) = \zeta_{3-\ell}$ so $g^{-1}_{\beta_\ell}(\zeta_{3-\ell}) = \beta_0$ for $\ell = 1,2$ hence $g^{-1}_{\beta_1}(\zeta_2) = g^{-1}_{\beta_2}(\zeta_1)$. As $c_5\{\alpha_1,\alpha_2\} = c_5\{\beta_1,\beta_2\}$ we have $c'_7\{\alpha_1,\alpha_2\} = c'_7\{\beta_1,\beta_2\}$ but $\zeta_1,\zeta_2 \le g_{\alpha_2}(\alpha_1)$ hence

$\circledast_3$ $(g^{-1}_{\alpha_\ell}(\zeta_1) < g^{-1}_{\alpha_\ell}(\zeta_2)) \equiv (g^{-1}_{\beta_\ell}(\zeta_1) < g^{-1}_{\beta_\ell}(\zeta_2))$ for $\ell = 1,2$.

As $\zeta_1 \ne \zeta_2$ we have $g^{-1}_{\alpha_1}(\zeta_1) \ne g^{-1}_{\alpha_1}(\zeta_2)$.

By symmetry without loss of generality $\zeta_1 > \zeta_2$ so $g^{-1}_{\beta_1}(\zeta_1) < g^{-1}_{\beta_1}(\zeta_2)$ <u>iff</u> (by equalities above) $g^{-1}_{\beta_2}(\zeta_2) < g^{-1}_{\beta_2}(\zeta_1)$ <u>iff</u> (the equivalence in $\circledast_3$) $g^{-1}_{\alpha_2}(\zeta_2) < g^{-1}_{\alpha_2}(\zeta_1)$ <u>iff</u> by the choice of $\zeta_1, g^{-1}_{\alpha_2}(\zeta_1) = \alpha_0), g^{-1}_{\alpha_2}(\zeta_2) < \alpha_0$ <u>iff</u> (as $c'_5\{\alpha_0,\alpha_2\} = c'_5\{\beta_0,\beta_1\}$ and $\zeta_2 < \zeta_1 = g_{\alpha_1}(\beta_0)), g^{-1}_{\beta_1}(\zeta_2) < \beta_0$ <u>iff</u> (as $\beta_0 = g^{-1}_{\beta_1}(\zeta_1)), g^{-1}_{\beta_1}(\zeta_2) < g^{-1}_{\beta_1}(\zeta_1)$, clear contradiction.

So we have proved $\boxdot_5$.

We can now sum up, i.e.:

Proof of 1.1(2) from $\boxdot_1 - \boxdot_5$. We are given $c_0 : [\mu^+]^{<\aleph_0} \to \mu$. First we apply $\boxdot_2$ for $c = c_0$ and get $c_2 : [\mu^+]^2 \to \mu$ as there.

Second, we apply $\boxdot_5$ for $c = c_2$ and get c_5 as there. Let us check that c_5 is as required on c^* in 1.1(2). So assume $(*)_0 + (*)_1$ below and (as the case $n = 2$ is trivial) assume $n \ge 3$ where

$(*)_0$ $\{\alpha_0,\dots,\alpha_{n-1}\} \in [\mu^+]^n$ and $\{\beta_0,\dots,\beta_{n-1}\} \in [\mu^+]^n$ and

$(*)_1$ $\ell < k < n \Rightarrow c_5\{\alpha_\ell,\alpha_k\} = c_5\{\beta_\ell,\beta_k\}$.

Without loss of generality (by renaming)

$(*)_2$ $\alpha_0 < \ldots < \alpha_{n-1}$.

and it is enough to prove that $c_0\{\alpha_0,\dots,\alpha_{n-1}\} = c_0\{\beta_0,\dots,\beta_{n-1}\}$. By clause (a) of $\boxdot_5$ we have

$(*)_3$ $\ell < k < n \Rightarrow c_2\{\alpha_\ell,\alpha_k\} = c_2\{\beta_\ell,\beta_k\}$.

By clause (c) of $\boxdot_5$ we have

$(*)_4$ $\ell < k < n \Rightarrow c_4\{\alpha_\ell,\alpha_k\} = c_4\{\beta_\ell,\beta_k\}$.

Hence by $\boxdot_4$ we have

$(*)_5$ if $\ell < k < n$ and $\ell < n-2$ then $\beta_\ell < \beta_k$.

[Why? Apply $\boxdot_4$ to $\alpha_\ell, \alpha_{\ell+1}, \alpha_k; \beta_\ell, \beta_{\ell+1}, \beta_k$ if $\ell+1 < k$, and apply $\boxdot_4$ to $\alpha_\ell, \alpha_{\ell+1}, \alpha_{\ell+2}; \beta_\ell, \beta_{\ell+1}, \beta_{\ell+2}$ if $\ell+1 = k$.]
So

$(*)_6(i)$ $\beta_0 < \beta_1 < \ldots < \beta_{n-3} < \beta_{n-2} < \beta_{n-1}$ or
(ii) $\beta_0 < \beta_1 < \ldots < \beta_{n-3} < \beta_{n-1} < \beta_{n-2}$.

So clause (β) of 1.1 holds.

If (i) of $(*)_6$ holds, then the choice of c_2, i.e., by $\boxdot_2$ and $(*)_3$ above we get $c_0\{\alpha_0, \ldots, \alpha_{n-1}\} = c_0\{\beta_0, \ldots, \beta_{n-1}\}$ so we are done. Otherwise we have (ii) of $(*)_6$ so by clause (b) of $\boxdot_5$ we have

$(*)_7$ if $\ell < n-2$ then $f_{\alpha_{n-1}}(\alpha_\ell) = f_{\beta_{n-2}}(\beta_\ell)$.

[Why? Apply $\boxdot_5(b)$ to $\alpha_\ell, \alpha_{n-2}, \alpha_{n-1}; \beta_\ell, \beta_{n-2}, \beta_{n-1}$.]
So by $\boxdot_3$ we get $c_0\{\alpha_0, \ldots, \alpha_{n-1}\} = c_0\{\beta_0, \ldots, \beta_{n-1}\}$ finishing. $\square_{1.1}$

1.3 Claim. *Defining* $\mathrm{ID}(\lambda, \mu)$, *we can restrict ourselves to* $c : [\lambda]^{<\aleph_0} \to \mu$ *such that* $c \restriction [\lambda]^1$ *is constant* *<u>if</u>* $\mathrm{cf}(\lambda) > \mu$.

1.4 Claim. *1) Assume* $\mu = \mu^{<\mu}$ *and* $n \in [1, \omega)$. *The identities of* $\mathrm{ID}(\mu^{+n}, \mu)$ *are* $(n+1)$*-simple (and also* $\mathrm{OID}(\mu^+, \mu)$*).*

Proof. As in 1.1, only easier in the additional cases. $\square_{2.1}$

§2 Successor of strong limit above supercompact: 2-identities

So we know that if μ is strong limit singular and there is a compact cardinal in $(\mathrm{cf}(\mu), \mu)$ then $\mathrm{ID}_2(\mu^+, \mu) \neq \mathrm{ID}_2(\aleph_1, \aleph_0)$. It seems desirable to find explicitly such 2-identity.

The proof of the following does much more.

2.1 Claim. *Assume*

(a) $\mathbf{s}_k = (k + \binom{k}{2}, e_{\mathbf{s}_k})$ *where the non-singleton* $e_{\mathbf{s}_k}$*-equivalence classes are the set sets here* $\binom{1}{2} = 0$*)*
$\{\{\ell_0, \ell_2\} : \ell_0 < k$ *and for some* $\ell_1 \in \{\ell_0 + 1, \dots, k-1\}$ *we have* $\ell_2 = k + \binom{\ell_1}{2} + \ell_0\}$ *and*
$\{\{\ell_1, \ell_2\} : \ell_1 < k$ *and for some* $\ell_0 < \ell_1$ *we have* $\ell_2 = k + \binom{\ell_1}{2} + \ell_0\}$

(b) μ *is strong limit,* θ *a compact cardinal and* $\mathrm{cf}(\mu) < \theta < \mu$.

1) $\mathbf{s}_k \in \mathrm{ID}_2(\mu^+, \mu)$*, moreover* $\mathbf{s}_k \in \mathrm{OID}_2(\mu^+, \mu)$.
2) $\mathbf{s}_k \notin \mathrm{ID}_2(\aleph_1, \aleph_0)$ *for* $k \geq 3$ *so for* $k = 3$ *we have* $\mathbf{s}_k = (6, e_{\mathbf{s}})$ *and the non-singleton equivalence classes, after permuting* $\{3, 5\}$ *are* $\{\{1,3\}, \{0,4\}, \{0,5\}\}$ *and* $\{\{1,5\}, \{2,3\}, \{2,4\}\}$.

Proof. Part (1) follows from subclaim 2.2(3) below and part (2) follows from 2.3 below. $\square_{2.1}$

2.2 Claim. *Assume*

(a) μ *is strong limit,*

(b) θ *is compact and* $\mathrm{cf}(\mu) < \theta < \mu$

(c) $\kappa = \mathrm{cf}(\mu), \langle \lambda_i : i < \kappa \rangle$ *is increasing with limit* μ

(d) $c : [\mu^+]^2 \to \mu$

(e) $d\{\alpha, \beta\} = \mathrm{Min}\{i : c\{\alpha, \beta\} < \lambda_i\}$.

1) We can find $i(*), A, f$ *such that*

$(*)(i)$ $i(*) < \kappa, A \in [\mu^+]^{\mu^+}$ *and* $f : A \to \lambda_{i(*)}$

(ii) *for every set* $B \subseteq A$ *of cardinality* $< \theta$ *there are* μ^+ *ordinals* $\gamma \in A$ *satisfying* $(\forall \alpha \in B)[d\{\alpha, \gamma\} = f(\alpha)]$.

2) In part (1) we also have: if $A_1 \subseteq A, |A_1| \ge \beth_n(\lambda)^+$ and $\lambda_{i(*)} \le \lambda < \mu$, <u>then</u> for some $\langle \gamma_\ell : \ell < n \rangle \in {}^n(\lambda_{i(*)})$ and $B \in [A_1]^\lambda$ for every $\alpha_0 < \ldots < \alpha_{n-1}$ from B for arbitrarily large $\beta < \lambda$ we have $\ell < n \Rightarrow c\{\alpha_\ell, \beta\} = \gamma_\ell$.
3) $\mathbf{s}_k \in \mathrm{ID}_2(c)$ where $\mathbf{s}_k$ is from clause (a) of 2.1.

Proof. 1) Let D be a uniform θ-complete ultrafilter on μ^+.

Define $f : \mu^+ \to \kappa$ by $f(\alpha) = i \Leftrightarrow \{\gamma < \mu^+ : d\{\alpha, \gamma\} = i\} \in D$, note that the function f is well defined as D is a θ-complete ultrafilter on μ^+ and $\theta > \kappa$. So for some $i(*)$, the set $A =: \{\alpha < \mu^+ : f(\alpha) = i(*)\}$ belongs to D and check that $(*)$ holds, that is (i) + (ii) hold.
2) Define $c^* : [A]^n \to {}^n(\lambda_{i(*)})$ such that

- $\circledast$ if $\alpha_0 < \ldots < \alpha_{n-1}$ are from A then for μ^+ ordinals $\beta < \mu^+$ we have $\langle c\{\alpha_\ell, \beta\} : \ell < n\}\rangle = c^*\{\alpha_0, \ldots, \alpha_{n-1}\}$.

So $\mathrm{Rang}(c^*)$ has cardinality $\le (\lambda_{i(*)})^n = \lambda_{i(*)}$ hence by the Erdös-Rado theorem there is $B \subseteq A_1$ infinite (even of any pregiven cardinality $< \lambda$) such that $c^* \restriction [B]^n$ is constant.
3) Straight: in part (2) use $n = 2, A_1 = A$ and get B and $\langle \gamma_0, \gamma_1 \rangle \in {}^2(\lambda_{i(*)})$ as there and choose $\alpha_0 < \ldots < \alpha_{k-1}$ from B. Next choose α_ℓ for $\ell = 0, 1, \ldots, \binom{k}{2} - 1$, choosing β_ℓ by induction on ℓ. If $\ell = \binom{\ell_1}{2} + \ell_0$ and $\ell_0 < \ell_1 < k$ choose $\beta_\ell \in A$ satisfying $\beta_\ell > \alpha_{k-1}$ and $\beta_\ell > \beta_m$ for $m < \ell$ such that $c\{\alpha_{\ell_0}, \beta_\ell\} = \gamma_0, c\{\alpha_{\ell_1}, \beta_\ell\} = \gamma_1$.
Now let $\alpha_{k+\ell} = \beta_\ell$ for $\ell < \binom{k}{2}$, and clearly $\langle \alpha_\ell : \ell < k + \binom{k}{2} \rangle$ realize the identity $\mathbf{s}_k$. $\square_{2.2}$

2.3 Subclaim. *1) If $\mathbf{s} \in \mathrm{ID}_2(\aleph_1, \aleph_0)$, <u>then</u> we can find a function $h : [\mathrm{Dom}_{\mathbf{s}}]^2/\mathbf{s} \to \omega$ respecting $e_{\mathbf{s}}$ (i.e. $\{\ell_1, \ell_2\} e_{\mathbf{s}} \{\ell_3, \ell_4\} \Rightarrow h\{\ell_1, \ell_2\} = h\{\ell_3, \ell_4\}$) and there is a linear order $<$ of $\mathrm{Dom}_{\mathbf{s}}$ satisfying*

- $\circledast$ *for any equivalence class $\mathbf{a}$ of e there are a_0, a_1 such that*
  - *(i) a_0, a_1 are disjoint finite subsets of $\mathrm{Dom}_{\mathbf{s}}$*
  - *(ii) if $\{\ell_0, \ell_1\} \in \mathbf{a}$ and $\ell_0 < \ell_1$ then $\ell_0 \in a_0$ & $\ell_1 \in a_1$*
  - *(iii) if $\ell_0 \ne \ell_1$ are from $a_0 \cup a_1$ and $\{\ell_0, \ell_1\} \notin \mathbf{a}$ <u>then</u> $h(\{\ell_0, \ell_1\}) > h(\mathbf{a})$.*

2) We can add in $\circledast$

- *(iv) if $\mathbf{a}_0, \mathbf{a}_1$ are distinct $\mathbf{e}_{\mathbf{s}}$-equivalence classes <u>then</u> for some $m \in \{0, 1\}$ we have $[\cup \mathbf{a}_m]^2 \backslash \mathbf{a}_m$ is disjoint to $\mathbf{a}_{1-m}$*

(v) *in* $\circledast$ *above* a_0, a_1 *can be defined as* $\{\ell_0 : \{\ell_0, \ell_1\} \in \mathbf{a}, \ell_0 < \ell_1\}, \{\ell_1 : \{\ell_0, \ell_1\} \in \mathbf{a}, \ell_0 < \ell_1\}$ *respectively.*

3) If $k \geq 3, \mathbf{s}_k$ *from 2.1 clause (a)* <u>*then*</u> $\mathbf{s}_k$ *does not belong to* $\mathrm{ID}_2(\aleph_1, \aleph_0)$.

Proof. 1) Remember that by 0.6 we can deal with $\mathrm{OID}(\aleph_1, \aleph_0)$. By [Sh 74] we know what is $\mathrm{ID}(\aleph_1, \aleph_0)$, i.e., the family of identities in $\mathrm{OID}(\aleph_1, \aleph_0)$ is generated by two operations; one is called duplication and the other of restriction (see below) from the trivial identity (i.e. $|\mathrm{dom}_{\mathbf{s}}| = 1$) and we prove $\circledast$ by induction on n, the number of times we need to apply the operations.
Recall that (a, e) is gotten by duplication if we can find sets a_0, a_1, a_2 and a function g such that

$\circledast_1(a)$ $a_0 < a_1 < a_2$ (i.e. $\ell_0 \in a_0, \ell_1 \in a_1, \ell_2 \in a_2 \Rightarrow \ell_0 < \ell_1 < \ell_2$)

(b) $a = a_0 \cup a_1 \cup a_2$

(c) g a one-to-one order preserving function from $a_0 \cup a_1$ onto $a_0 \cup a_1$ (so $g \restriction a_0 = \mathrm{id}_{a_0}$; let $g_1 = g, g_2 = g^{-1}$

(d) for $\ell_0 \neq \ell_1 \in (a_0 \cup a_1)$ we have $\{\ell_0, \ell_1\} e \{g(\ell_0), g(\ell_1)\}$

(e) if $\ell_1 \in a_1, \ell_2 \in a_2$ then $\{\ell_1, \ell_2\}/e$ is a singleton

(f) $\mathbf{s}_\ell = (a_0 \cup a_\ell, e \restriction [a_0 \cup a_\ell]^2)$ is from a lower level (up to isomorphism).

Recall that (a, e) is gotten by restriction from (a', e') if $a \subseteq a', e = e' \restriction [a]^2$.

Now we prove the existence of h as required by induction on the level. If $|\mathrm{Dom}_{\mathbf{s}}| = 1$ this is trivial. If $\mathbf{s}$ is gotten by restriction it is trivial too, (as if $\mathbf{s} = (a, e), s' = (a', e'), a' \subseteq a, e' = e \restriction a'$ and $h : [a]^2/e$ is as guaranteed then we let $h'(\{\ell_0, \ell_1\}/e') = h(\{\ell_0, \ell_1\}/e)$ for $\ell_0 < \ell_1$. Easily h' is as required). So assume $\mathbf{s} = (a, e)$ is gotten by duplication, so let a_0, a_1, a_2, g_1, g_2 be as in $\circledast_1$ and let h_1 be as required for $\mathbf{s}_1 = (a_0 \cup a_1, e \restriction [a_0 \cup a_1)^2)$ and similarly define h_2 by $h_2\{\alpha, \beta\} = h_1\{g_2(\alpha), g_2(\beta)\}$. Let $n^* = \max \mathrm{Rang}(h_1)$ and define $h : [a_0 \cup a_1 \cup a_2]^2 \Rightarrow \omega$ by $h \supseteq h_1, h \supseteq h_2$ and if $k \in a_1, \ell \in a_2$ then we let $h\{k, \ell\} = n^* + 1$. Now check.
2) By symmetry, without loss of generality $h(\mathbf{a}_0) < h(\mathbf{a}_1)$ and now $m = 1$ satisfies the requirement by applying $\circledast_1$ to the equivalence class $\mathbf{a} = \mathbf{a}_1$.
3) It is enough to deal with $\mathbf{s}_3$. By direct checking the criterion in part (2) fails.
$\square_{2.3}$

The following is like 2.1 with μ just limit (not necessarily a strong limit cardinal) so

2.4 Claim. *Assume*

(*a*) $\mathbf{s}'_n \in \mathrm{OID}_2$ *is* $(2n+n^2, e_{\mathbf{s}'_n})$ *where the non-singleton* $e_{\mathbf{s}'_n}$*-equivalence classes are*
$\{\{\ell_0, 2n+n\ell_0+\ell_1\} : \ell_0, \ell_1 < n\}$ *and*
$\{\{n+\ell_1, 2n+n\ell_0+\ell_1\} : \ell_0, \ell_1 < n\}$

(*b*) μ *is a limit cardinal,* $\mu > \theta > \mathrm{cf}(\mu)$ *and* θ *is a compact cardinal*

(*c*) $s''_n \in \mathrm{OID}_n$ *is* $(2^n + 2^{2n}, e_{\mathbf{s}''_s})$ *where the non-singleton* $e_{\mathbf{s}''_n}$*-equivalence classes are: for* $m < n, \eta \in {}^m 2, i = 0, 1$ *let* $\mathbf{a}^i_\eta = \{\{\ell_i, 2^n + \binom{2^n}{\ell_0} + \ell_1\} : \ell_0, \ell_1 < 2^n$ *and for some* $\nu_0, \nu_1 \in {}^n 2$ *we have* $\eta\hat{\ }\langle 0\rangle \trianglelefteq \nu_0, \eta\hat{\ }\langle 1\rangle \trianglelefteq \nu_1$ *and* $\ell_0 = \Sigma\{\nu_0(j)2^j : j < n\}$ *and* $\ell_1 = \Sigma\{\nu_1(j)2^j : j < n\}\}$.

1) $\mathbf{s}'_n \in \mathrm{ID}_2(\mu^+, \mu)$*, moreover* $\mathbf{s}'_n \in \mathrm{OID}_2(\mu^+, \mu)$ *similarly for* $\mathbf{s}'_n$.
2) $\mathbf{s}'_n \notin \mathrm{ID}_2(\aleph_1, \aleph_0)$ *for* $n \geq 2$*, similarly for* $\mathbf{s}''_n$.

Proof. 1) Like the proof of 2.2 using [Sh 49] (or just [Sh 604, §5]) instead of the Erdös-Rado theorem.
2) Otherwise there is $(a, e) \in \mathrm{ID}_2(\aleph_1, \aleph_0)$ and an embedding h of $\mathbf{s}'_n$ into (a, e) and by 0.6 without loss of generality h is order preserving and $(a, e) \in \mathrm{OID}_2(\aleph_1, \aleph_0)$. Now

$(*)_1$ if $\ell_0 < n, \ell_1 < n$ and $\ell = 2n + n\ell_0 + \ell_1$ <u>then</u> $h(\ell_0) < h(\ell)$.
[Why? Choose $\ell'_1 < n, \ell'_1 \neq \ell_1$ and $\ell' = 2n + n\ell_0 + \ell'_1$, so $\ell \neq \ell'$ and $\{\ell_0, \ell\} e_{\mathbf{s}'_n} \{\ell_0, n + \ell'\}$ hence $\{h(\ell_0), h(\ell)\}, \{h(\ell_0), h(\ell')\}$ are e-equivalent and $h(\ell) \neq h(\ell')$. But on (a, e) we know that if $\{m_0, m_1\} e \{m_0, m_2\}$ then $m_2 < m_1 < m_0$ and $m_2 < m_0 < m_1$ are impossible (see 2.5(2) below) so we are done.]

$(*)_2$ if $\ell_0 < n, \ell_1 < n$ and $\ell = 2n + n\ell_0 + \ell_1$ then $h(\ell_1) < h(\ell)$.
[Why? Like $(*)_1$.]

Now we apply 2.3(1) + (2) above so $\mathbf{s}'_n \notin \mathrm{ID}_2(\aleph_2, \aleph_1)$. The conclusion about $\mathbf{s}''_n$ follows. $\square_{2.4}$

2.5 Observation. 1) If $k \geq 2, \mathbf{s} = (n, e) \in \mathrm{OID}_2(\mu^+, \mu)$ <u>then</u> we can find $\mathbf{s}' = (n', e')$ in fact $n' = 2n - 1$ such that:

(*i*) $e' \restriction [n]^2 = e$

(*ii*) $\mathbf{s}' \in \mathrm{ID}(\mu^+, \mu)$

(*iii*) for every $c : [\mu^+]^{<\aleph_0} \to \mu$ there is $c' : [\mu^+]^{<\aleph_0} \to \mu$ refining c (i.e. $c'(u_1) = c'(u_2) \Rightarrow c(u_1) = c(u_2)$) such that: if $h : \{0, \ldots, 2n-2\} \to \mu^+$ is one to

one and satisfies $u_1 e' u_2 \Rightarrow c'(h''(u_1)) = c'(h''(u_2))$ then $h \restriction \{0, \dots, n-1\}$ is increasing.

2) There is $c : [\mu^+]^2 \to \mu$ such that:
if α, β, γ are distinct and $c\{\alpha, \beta\} = c\{\alpha, \gamma\}$ then $\alpha < \beta \ \& \ \alpha < \gamma$.
3) We can replace in (1), (μ^+, μ) by (λ, μ) if there is $\mathbf{s} = (n, e) \in \mathrm{ID}(\lambda, \mu)$ such that for some $c : [\lambda]^{<\aleph_0} \to \mu$ such that

$\circledast$ if $h : n \to \lambda$ induces $e_{\mathbf{s}}$ then $h(0) < h(1)$.

Proof. 1) Define e' : $u_1 e' u_2$ iff $u_1 e u_2 \vee u_1 = u_2 \vee \bigvee_{\ell < n-1} (u_1 = \{\ell, n+\ell+1\} \ \& \ u_2 e \{\ell, \ell+1\}) \vee \bigvee_{\ell < n} (u_2 = \{\ell, n+\ell+1\} \ \& \ u_1 e \{\ell, \ell+1\})$. Now use (2).
2) Let $f_\alpha : \alpha \to \mu$ be one to one and let $<^*$ a dense linear order on μ^+ with $\{\alpha : \alpha < \mu\}$ a dense subset. Now choose $c_1 : [\mu^+]^2 \to \mu$ such that $\alpha < \beta \Rightarrow \alpha \leq^* c_1\{\alpha, \beta\} <^* \beta$ and $c : [\mu^+]^2 \to \mu$ be $\alpha < \beta \Rightarrow c\{\alpha, \beta\} = \mathrm{pr}(f_\beta(\alpha), c_1\{\alpha, \beta\})$ for some pairing function pr.
3) Similar to part (1) only $|\mathrm{Dom}_{\mathbf{s}'}|$ is larger. $\square_{2.5}$

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